Spectral statistics for the evolution operator of a quantum particle showing chaotic diffusion of the coordinate

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We study the spectral properties of the evolution operator of a quantum particle subject to a space-periodic time-dependent potential. Two qualitatively different regimes of the system dynamics are compared: case (i), when the spreading of the wave packet is asymptotically ballistic; and case (ii), when the wave packet spreads diffusively. As time increases, the spectrum is shown to approach Poisson statistics in case (i) and circular unitary ensemble statistics in case (ii). A scaling relation for the velocity and curvature distributions of the spectral bands are found. [S1063-651X(97)12907-5]

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This paper continues our study of chaotic diffusion of a quantum particle in a time-periodic and space-periodic potential [1]. It should be noted that in the past few years this problem changed its status from a problem of academic interest to that of considerable experimental interest [2]. In the experiments cited dynamics of the ultra-cooled atoms subject to a standing laser wave with amplitude modulated in time has been studied. In the classical approach this system is governed by the Hamiltonian

$$H=\frac{p^2}{2m}+V(t)\cos^2(k_w x), \qquad (1)$$

where m is the mass of atom, k_w is the wave vector of a standing wave, $V(t)$ is proportional to laser intensity and, depending on the way we modulate the laser intensity, can exhibit chaotic diffusion both of the momentum and coordinate. The later case can be realized, for example, by choosing $V(t)\sim\sin^2(\nu t)$ (the detailed classical analysis of this system is given in [3]) or $V(t)$ to be a square-shape periodic function [4].

In this present paper we address the question of how the process of chaotic diffusion of the coordinate relates to statistics of the spectrum of the system evolution operator. Two preliminary remarks should be of interest. First, since our system is a system with translational symmetry, its spectrum has a band structure [5]. Thus we do statistics of the bands, where the Bloch vector is considered as a parameter. The second remark concerns the problem of separation of the spectrum into regular and chaotic components. This problem occurs because the phase space of any physical system of type (1), with $V(t)$ yielding diffusion of the coordinate, has two components—a chaotic component for $|p|<p_{max}$ (confinement of the momentum is a necessary condition for diffusion of the coordinate); and a regular component for $|p|>p_{max}$, where a classical particle moves ballistically. Thus the spectrum of the evolution operator is a mixture of the bands associated with ballistic and chaotic motion (see Fig. 3 in Ref. [1]). Separation of the spectrum into two components is a rather complicated numerical problem [11]. We note that the two-component structure of the phase space affects the classical dynamics as well. In particular, the classical diffusion has been found to be typically an anomalous (super-)diffusion [3].

One can avoid all the problems caused by the mixed structure of the system phase space by considering a model system. To this end we introduce the Harper model [6–10] with impulses

$$H=\cos p+K\tilde{\delta}(t)\cos x, \qquad \tilde{\delta}(t)=\sum_m \delta(t-Tm), \qquad (2)$$

where the momentum is confined by the interval $0\leqslant p<2\pi$ and $-\infty<x<\infty$. (We note that topology of the phase space differs from those considered in [6–10].) For $K\geqslant 5$ system (2) seems to have only a chaotic component (no stability islands are visible under the resolution $2\pi/100$) and shows a perfect diffusion in the classical approach [see Fig. 1, curve (a)]. Because the momentum is bounded, the diffusion constant D_{cl} is essentially independent of K and, neglecting a small oscillation, $D_{cl}(K)\approx 0.05$.

Quantization of model (2) is straightforward, $\cos p\to\cos(-i\hbar\partial/\partial x)$, and leads to the Harper-like equation

$$i\hbar\,\partial\psi(x_n,t)/\partial t=[\psi(x_{n+1},t)+\psi(x_{n-1},t)]/2 + K\tilde{\delta}(t)\cos x_n\psi(x_n,t). \qquad (3)$$

In Eq. (3) $x_n=\hbar n$ and, to satisfy the periodic boundary condition for the momentum, the scaled Planck constant $\hbar=2\pi/N$. We would like to stress that here we do not regard Eq. (3) as the tight-binding model of crystal electron in a magnetic field, but consider it as a convenient approximation of system (1). Thus the rationality of $\hbar/2\pi$, which preserves the translational symmetry of the system, is an important point in our model. In what follows we study the spectral properties of evolution operator of system (2) in two cases—case (i), when $\tilde{\delta}(t)$ is a periodic train of δ functions; and case (ii), when $\tilde{\delta}(t)$ is an aperiodic train of the impulses (we choose the interval between impulses to be either $T=1$ or $T=0.9$ in random sequence). These two cases are qualitatively different from the viewpoint of the system dynamics. In the first case the spreading of the wave packet (initially

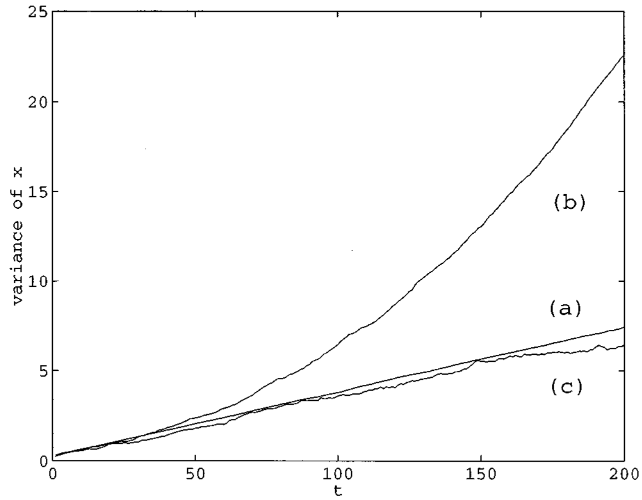

FIG. 1. Variance of the coordinate for the Harper model with impulses ($K=5$) as a function of time. Time is measured in the number of impulses, the coordinate in units of 2π. Curve (a) shows the classical diffusion, (b) is the quantum ''diffusion'' in case (i) [periodic train of impulses], and (c) is the quantum diffusion in case (ii) [aperiodic train of impulses]. The value of the scaled Planck constant is $\hbar=2\pi/64$. The initial condition is chosen in the form of a localized wave packet $|\psi(x,0)|^2 \sim \exp[-(x/\pi)^2]$. In the classical approach it corresponds to the Gauss distribution with the variance $\sigma_x^2=\pi^2$ and $\sigma_p^2=\hbar^2/4\sigma_x^2$.

well localized) is asymptotically ballistic, but the spreading of the packet is asymptotically diffusive in the second case [4]. Thus the system evolution operator is a banded matrix (in the coordinate representation) with the band width growing linearly in time in case (i) and as a square root of time in case (ii).

A few words about the numerical method used. Because an eigenfunction of the evolution operator is the Bloch function $\psi_l(x_n)=\exp(ikx_n)\phi_l(x_n)$, $\phi_l(x_n+2\pi)=\phi_l(x_n)$, the problem of finding the spectrum is reduced to diagonalization of the matrix $U(t)$

$$U(t)=\prod_{\tau=1}^{t} U_{T_\tau}, \tag{4}$$

of finite size $N\times N$. In Eq. (4) U_T is unitary matrix of the system evolution for one step

$$U_T=\exp\left(-\frac{iT}{\hbar}\cos(\hat{p}+k)\right)\exp\left(-\frac{iK}{\hbar}\cos x\right), \tag{5}$$

which is now defined on the torus $0\leqslant x,p<2\pi$, k is the Bloch vector. The matrices (5) and (4) were calculated in the momentum representation for every k and then diagonalized.

First we will discuss the spectral properties of the matrix (5). The spectrum of U_T is shown in Fig. 2(a) (the phases of the eigenvalues $\lambda=\exp[i\epsilon_l(k)]$ are plotted). It is seen that the spectrum is symmetric with respect to $k\rightarrow -k$ and $\epsilon\rightarrow-\epsilon$. However this symmetry does not influence statistics of the spectrum. A property which does influence statistics is the antiunitary symmetry of U_T for any fixed k [13]. Because of this the spectrum is expected to obey circular orthogonal ensemble (COE) rather than circular unitary ensemble (CUE) statistics. The solid line (a) in Fig. 3 shows the cumulative distribution function $I(s)=\int_0^s P(s')\mathrm{d}s'$ for the gaps

$$s=\frac{\epsilon_{l+1}(k)-\epsilon_l(k)}{\hbar}\approx\frac{1}{\hbar}\frac{\partial\epsilon_l(k)}{\partial l}. \tag{6}$$

[$P(s)ds$ is the probability to find a gap of the size s. We remind the reader that $P(s)=\exp(-s)$ for Poisson, $P(s)=(\pi/2)s\exp(-\pi s^2/4)$ for COE, and $P(s)=(32/\pi^2)s^2\exp(-4s^2/\pi)$ for CUE statistics.] A satisfactory agreement between the numerical data and the prediction of the random matrix theory (dotted line) is seen [14].

We proceed with the analysis of the spectrum of the evolution operator (4). We begin with case (i), when $T_\tau=1$. In this case the motion of a quantum particle in a space-periodic potential is asymptotically ballistic [1,4,12] and results in

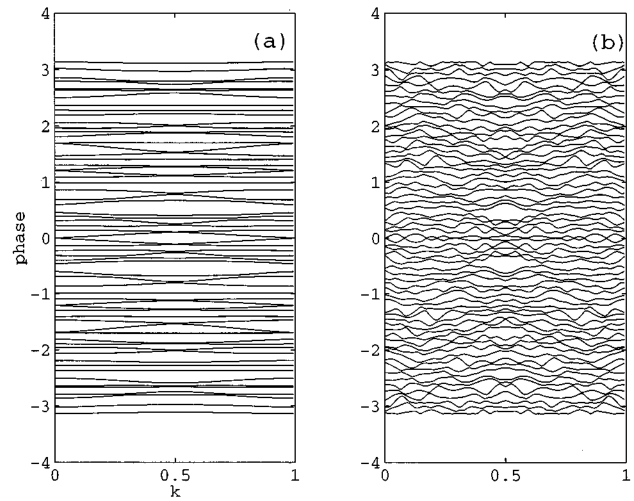

FIG. 2. Spectrum of the system evolution operator $U(t)$ for $t=1$ (a) and $t=100$ (b) in the case of aperiodic train of impulses.

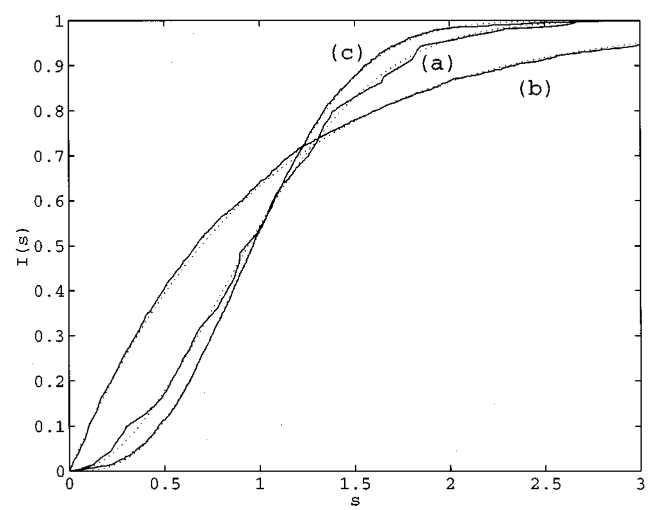

FIG. 3. Cumulative distribution function for the gaps. Statistics of the gaps for the matrix $U_T=U(1)$ [curve (a)] and for $U(t)$ at $t=100$ in case (i) [curve (b)] and case (ii) [curve (c)]. The statistics are done by collecting the eigenvalues of $U(t)$ for 256 equally spaced values of k. The dotted lines show Poisson, COE, and CUE statistics according to the random matrix theory.

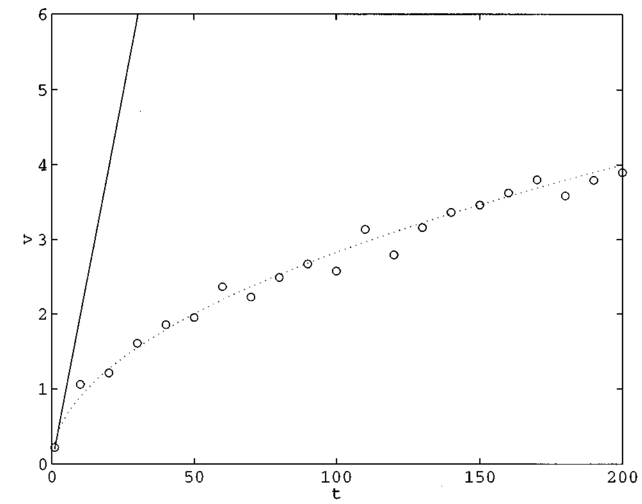

FIG. 4. Dependence of the characteristic velocity $\overline{v}$ (in units of $\hbar$) on time in the case of diffusive spreading (circles). The dotted line is the best fit by $\overline{v}(t) \sim t^{1/2}$. The solid line shows the growth of $\overline{v}$ in the case of ballistic spreading.

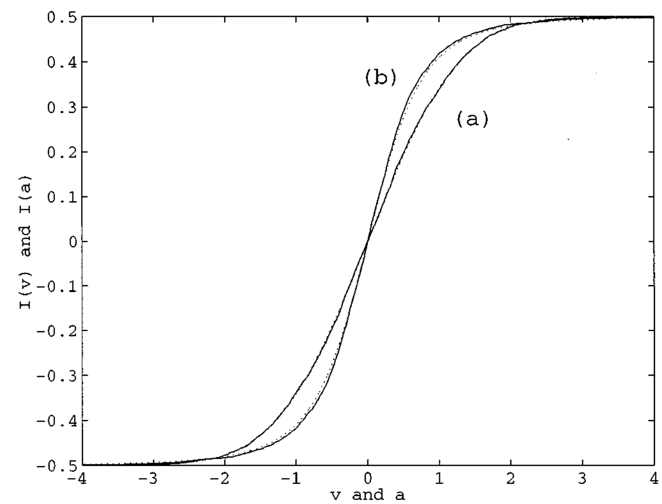

FIG. 5. Cumulative distribution function for the normalized velocity (a) and curvature (b) for the spectrum shown in Fig. 2(b).

t^2 law for variance of the coordinate $\langle x^2 \rangle \sim t^2$ [see Fig. 1, curve (b)]. This ballistic spreading is mirrored in the change of spectral statistics from COE to Poissonian. In fact, the eigenvalues of the evolution operator for any t are just $\lambda(t) = \exp(i\epsilon t)$. Thus the crossings of the bands are allowed, which lead to Poisson statistics [see Fig. 3, curve (b)]. We note that the transition from COE to Poisson occurs already for $t \approx 10$. This time differs from the quantum-classical crossover time $t_c \approx 40$, when one can see an apparent deviation in the behavior of $\langle x^2 \rangle$ in Fig. 1. Unfortunately, available computer facilities were not enough to study the question, how the crossover time for the spectrum relates to the crossover time for the system dynamics [15]. We plan to return to this problem in future.

We proceed with case (ii). It is seen from Fig. 1 that now the spreading of the wave packet is diffusive. A change of the system dynamics can be explained in the following way. Randomness introduced is actually equivalent to some external noise. It is well known that external noise recovers the classical dynamics of a semiclassical system [16] and the behavior of any quantum mean $\langle A \rangle_{qu} = \langle \psi(t) | \hat{A} | \psi(t) \rangle$, averaged over different realizations of a random process, coincides with the classical mean, i.e., $\overline{\langle A \rangle_{qu}} = \overline{\langle A \rangle_{cl}}$. While this is true for the average, this should also be true for a ''typical'' realization of a random process (i.e., for the case we are currently considering). The spectrum of the evolution operator for $t = 100$ is shown in Fig. 2(b) and cumulative distribution function of the gaps is given by curve (c) in Fig. 3. Since now the dynamics is not invariant with respect to time inversion, statistics has changed from COE to CUE. (The crossover time was again about 10.) From Fig. 3 it is also seen that statistics for $U(t)$ fits the prediction of the random matrix theory (RMT) much better than statistics for U_T. In particular, this is true for a relative large value of the scaled Planck constant. For example, for $\hbar = 2\pi/16$ the distribution of the gaps for U_T does not follow any smooth curve at all, but statistics for the evolution operator $U(t)$ rapidly approach either Poisson or CUE statistics when time tends to infinity.

Approximation of the spectrum by CUE statistics seems to be a general property of a system exhibiting diffusion. Another property of the distribution of the velocity can be found if we analyze

$$v = \frac{1}{\hbar} \frac{\partial \epsilon_l(k)}{\partial k}. \tag{7}$$

It is seen from Fig. 2 that in the course of time evolution the structure of the spectrum changes from ''laminar'' to ''turbulent.'' For the ''turbulent'' regime we have found a perfect correspondence of the velocity distribution to the Gauss distribution $P(v) = (2\pi)^{-1/2} \exp(-v^2/2\overline{v^2})$ [see Fig. 5, curve (a)] with monotonically growing $\overline{v}$. A functional dependence $\overline{v} = \overline{v}(t)$ can be guessed by considering case (i). In that case both the characteristic velocity and the region of support of the wave function grows linearly with time. Thus we can assume that in the present case (ii) the characteristic velocity grows as $\overline{v} \sim t^{1/2}$. The numerical data support this conjecture (see Fig. 4).

It is also interesting to study the distribution of the curvature of the bands. Usually the curvature distribution for the level flow of a chaotic system depends on the way of parametrization and only the asymptotic behavior for large curvature (the tail of the distribution) can be universal [17]. This leaves some freedom in choosing a parametrization, which would fit the prediction of RMT in the best way. For the system considered we have no such freedom—our parametrization by the Bloch vector k is defined by the physics of the problem, but not by our desire. We have found the distribution of the normalized curvature

$$a = \frac{1}{2\pi\hbar\beta\overline{v^2}} \frac{\partial^2 \epsilon_l(k)}{\partial k^2}, \quad \beta = 2 \tag{8}$$

to be invariant with respect to time and consistent with the prediction of RMT given by von Oppen's formula $P(a) = 2/\pi(1+a^2)^2$ [18] (see Fig. 5 curve [b]). We especially note that the correspondence of the numerical data to von Oppen formula is global.

In this paragraph we sum up the results obtained. We have considered a quantum particle moving in a space-periodic time-dependent potential, focusing on the case when the system exhibits chaotic diffusion of the coordinate. In this case the spectrum of the system evolution operator was shown to possess a number of properties. Namely: (a) the distribution of the gaps obeys CUE statistics; (b) the distribution of the velocity is Gaussian with the variance growing linearly in time; and (c) the distribution of the normalized curvature is independent of time and fitted by von Oppen's formula globally. Though these properties were found in a model system, we believe that they are universal for any chaotic system with translational symmetry, provided that only chaotic component of the spectrum is considered. A justification of this conjecture could be a subject of a different paper.

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